

## ON THE COMPACTIFICATION OF CONCAVE ENDS

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## 1. INTRODUCTION

**1.1. Definition.** We say that  $\rho : X \rightarrow ]a, b[$  is a **1-corona** if:

- $a \in \mathbb{R} \cup \{-\infty\}$ ,  $b \in \mathbb{R} \cup \{\infty\}$ ,
- $X$  is a complex manifold,
- $\rho$  is a strictly plurisubharmonic  $\mathcal{C}^\infty$ -function defined on  $X$  and with values in the open interval  $]a, b[$  such that the sets  $\{\alpha \leq \rho \leq \beta\}$ ,  $a < \alpha < \beta < b$ , are compact.

We say that the concave end of a 1-corona  $\rho : X \rightarrow ]a, b[$  can be **compactified** if  $X$  is (biholomorphic to) an open subset of a complex space  $\widehat{X}$  such that, for  $a < c < b$ , the set  $(\widehat{X} \setminus X) \cup \{a < \rho \leq c\}$  is compact.

The concave end of a 1-corona  $\rho : X \rightarrow ]a, b[$  always can be compactified if  $n := \dim X \geq 3$ . This was proved by Rossi [Ro] and Andreotti-Siu [AS]. For  $n = 2$  this is not true in general, as shown by a counterexample of Grauert, Andreotti-Siu and Rossi [AS, Gr, Ro]. However, if the concave end of a 1-corona  $\rho : X \rightarrow ]a, b[$  is even *hyperconcave* (i.e.  $a = -\infty$ ), then this is again true also for  $\dim X = 2$ . This was proved by Marinescu-Dinh [MD].

The Andreotti-Vesentini separation theorem [AV] implies the following necessary condition: If the concave end of a 1-corona  $\rho : X \rightarrow ]a, b[$  can be compactified,  $\dim X \geq 2$ , then  $H^1(X)$  is Hausdorff, i.e. the space of exact  $\mathcal{C}_{0,1}^\infty$ -forms is closed with respect to uniform convergence on compact sets together with all derivatives.

In the present paper we show that this condition is also sufficient. We prove:

**1.2. Theorem.** *Let  $\rho : X \rightarrow ]a, b[$  be a 1-corona such that  $H^1(X)$  is Hausdorff, and let  $n := \dim X \geq 2$ .*

*Then  $X$  can be embedded to  $\mathbb{C}^{2n+1}$ .*

*Hence, by the Harvey-Lawson theorem [HaLa], the concave end of  $X$  can be compactified.*

**1.3. Remark.** In view of the counterexample of Grauert, Andreotti-Siu and Rossi mentioned above, theorem 1.2 shows that theorem 2 in [Ra], which states that  $H^1(X)$  is Hausdorff for each 1-corona  $\rho : X \rightarrow ]a, b[$  also for  $\dim X = 2$ , cannot be true.

The present paper is a development of the PhD thesis of the first author [Br], where the following theorem is proved:

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**1.4. Theorem.** *Let  $\rho : X \rightarrow ]a, b[$  be a 1-corona,  $\dim X \geq 2$ . Assume, for some  $a < t < b$ ,  $H^1(\{a < \rho < t\})$  is Hausdorff. Then there exists  $\varepsilon > 0$  such that  $\{t - \varepsilon < \rho < b\}$  can be embedded to some  $\mathbb{C}^N$ .*

Theorem 1.4 can be deduced as follows from theorem 1.2: Suppose the hypotheses of theorem 1.4 are satisfied. Then, by theorem 1.2,  $\{a < \rho < t\}$  can be embedded to some  $\mathbb{C}^N$ . By the Harvey-Lawson theorem this implies that the concave end of  $\{a < \rho < t\}$  and, hence, the concave end of  $X$  can be compactified. By the Andreotti-Grauert theorem this further implies that  $H^1(X)$  is Hausdorff. Again using theorem 1.2, we conclude that  $X$  (and in particular  $\{t - \varepsilon < \rho < b\}$ ) can be embedded to some  $\mathbb{C}^N$ . Note however that in [Br] a direct proof of theorem 1.4 is given, not using the compactification of the concave end.

The new contribution of the present paper is the observation that if  $\rho : X \rightarrow ]a, b[$  is a 1-corona such that  $H^1(X)$  is Hausdorff, then  $H^1(\{a < \rho < t\})$  is Hausdorff for all  $a < t < b$  (see theorem 2.5 below). After that we could continue as in [Br] to prove the following two assertions on the existence of "many" holomorphic functions:

- (A) The global holomorphic functions on  $X$  separate points.
- (B) For each  $z \in X$  there exist  $n$  global holomorphic functions on  $X$  which form a coordinate system at  $z$ .

From (A) and (B), the claim of theorem 1.2 then follows in the same way as in the proof of the embedding theorem for Stein manifolds (see, e.g., theorem 5.3.6 in [Ho]).

However, to prove (A) and (B) as in [Br], boundary estimates for the  $\bar{\partial}$ -equation are needed, which are well-known but nevertheless quite technical. To show that this is not necessary, in the present paper we give another proof using only inner estimates for the  $\bar{\partial}$ -equation. A similar construction can be found already in [HL2] (section 23), where assertions (A) and (B) are deduced from certain stronger hypothesis, which implies the Hausdorffness of  $H^1(\{\rho < t\})$  for each  $a < t < b$ .

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## 2. EACH $H^1(\{\rho < t\})$ IS HAUSDORFF IF $H^1(X)$ IS HAUSDORFF

**2.1. Definition.** Let  $Y$  be a complex manifold of complex dimension  $n$ .

Then we denote by  $\mathcal{E}^{p,q}(Y)$ ,  $0 \leq p, q \leq n$ , the Fréchet space of  $\mathcal{C}_{p,q}^\infty$ -forms on  $Y$  endowed with the topology of uniform convergence together with all derivatives on the compact subsets of  $Y$ .

If  $K \subseteq Y$  is a compact set, then we denote by  $\mathcal{D}_K^{p,q}(Y)$  the subspace of  $\mathcal{E}^{p,q}(Y)$  which consists of the forms with support in  $K$ , endowed with the same topology.

By  $\mathcal{D}^{p,q}(Y)$  we denote the space all  $\mathcal{C}_{p,q}^\infty$ -forms on  $Y$  with compact support, endowed with the Schwartz topology, i.e. the finest local convex topology such that, for each compact set  $K \subseteq Y$ , the embedding  $\mathcal{D}_K^{p,q}(Y) \rightarrow \mathcal{D}^{p,q}(Y)$  is continuous.

Recall the following well known fact from Serre duality [L]:

**2.2. Proposition.** *Let  $Y$  be a complex manifold of dimension  $n$ . Then the space  $\bar{\partial}\mathcal{E}^{0,0}(Y)$  is closed in  $\mathcal{E}^{0,1}(Y)$ , if and only if,  $\bar{\partial}\mathcal{D}^{n,n-1}(Y)$  is closed in  $\mathcal{D}^{n,n}(Y)$ .*

We need also the following supplement to this (see theorem 2.7. in [LL]):

**2.3. Proposition.** *Let  $Y$  be a complex manifold of dimension  $n$ . Then the space  $\bar{\partial}\mathcal{D}^{n,n-1}(Y)$  is closed in  $\mathcal{D}^{n,n}(Y)$ , if and only if, for each compact set  $K \subseteq Y$ , the space  $\mathcal{D}_K^{n,n}(Y) \cap \bar{\partial}\mathcal{D}^{n,n-1}(Y)$  is closed in  $\mathcal{D}_K^{n,n}(Y)$ .*

Moreover, we shall use the following result from Andreotti-Grauert theory (see, e.g., theorem 16.1 in [HL2]):

**2.4. Proposition.** *Let  $\rho : X \rightarrow ]a, b[$  be a 1-corona, and let  $a < c < b$ . Then each  $\bar{\partial}$ -closed  $f \in \mathcal{E}^{n,n-1}(X)$  with  $f \equiv 0$  on  $\{c < \rho < b\}$  is  $\bar{\partial}$ -exact.*

Using these propositions, we now obtain:

**2.5. Theorem.** *Let  $\rho : X \rightarrow ]a, b[$  be a 1-corona such that  $H^1(X)$  is Hausdorff. Then  $H^1(\{\rho < t\})$  is Hausdorff for all  $a < t < b$ .*

*Proof.* Let  $a < t < b$ , and let  $K \subseteq \{\rho < t\}$  be a compact set. By propositions 2.2 and 2.3, it is sufficient to prove that the space  $\mathcal{D}_K^{n,n}(\{\rho < t\}) \cap \bar{\partial}\mathcal{D}^{n,n-1}(\{\rho < t\})$  is closed in the Fréchet space  $\mathcal{D}_K^{n,n}(\{\rho < t\})$ .

Let a sequence  $f_\nu \in \mathcal{D}_K^{n,n}(\{\rho < t\}) \cap \bar{\partial}\mathcal{D}^{n,n-1}(\{\rho < t\})$  be given which converges to some  $f \in \mathcal{D}_K^{n,n}(\{\rho < t\})$  with respect to the topology of  $\mathcal{D}_K^{n,n}(\{\rho < t\})$ . We have to find  $w \in \mathcal{D}^{n,n-1}(\{\rho < t\})$  with  $\bar{\partial}w = f$ .

Extending by zero, we may view  $f_\nu$  as forms in  $\bar{\partial}\mathcal{D}^{n,n-1}(X)$ , and  $f$  as a form in  $\mathcal{D}^{n,n}(X)$ . As  $H^1(X)$  is Hausdorff it follows from proposition 2.2 that  $\bar{\partial}\mathcal{D}^{n,n-1}(X)$  is closed in  $\mathcal{D}^{n,n}(X)$  with respect to the Schwartz topology. Moreover, as all  $f_\nu$  belong to  $\bar{\partial}\mathcal{D}^{n,n-1}(X)$  and the embedding  $\mathcal{D}_K^{n,n}(X) \rightarrow \mathcal{D}^{n,n}(X)$  is continuous (by definition of the Schwartz topology),  $f_\nu$  converges to  $f$  with respect to the Schwartz topology of  $\mathcal{D}^{n,n}(X)$ . Hence  $f \in \bar{\partial}\mathcal{D}^{n,n-1}(X)$ , i.e.  $f = \bar{\partial}u$  for some  $u \in \mathcal{D}^{n,n-1}(X)$ .

Choose  $a < s_f < s'_f < t$  with  $\text{supp } f \subseteq K \subseteq \{s_f < \rho < s'_f\}$ . Moreover, as  $u$  has compact support in  $X$ , we can find  $a < s_u < s_f < t < s'_u < b$  such that  $\text{supp } u \subseteq \{s_u < \rho < s'_u\}$ . Then  $\bar{\partial}u = f = 0$  on  $\{s'_f < \rho\}$ , and, since  $\rho : \{s'_f < \rho\} \rightarrow ]s'_f, b[$  is a 1-corona, this implies by proposition 2.4 that  $u|_{\{s'_f < \rho\}} = \bar{\partial}v$  for some  $v \in \mathcal{E}^{n,n-2}(\{s'_f < \rho\})$ . Choose  $s'_f < t' < t$  and a  $\mathcal{C}^\infty$ -function  $\chi$  on  $X$  such that  $\chi \equiv 1$  on  $\{t' \leq \rho\}$  and  $\chi \equiv 0$  on  $\{\rho \leq s'_f\}$ . Set

$$w = \begin{cases} u & \text{on } \{\rho \leq t'\}, \\ u - \bar{\partial}(\chi v) & \text{on } \{t' < \rho\}. \end{cases}$$

Then  $w \in \mathcal{D}_{\{s'_u \leq \rho \leq t'\}}^{n,n-1}(X)$  and  $\bar{\partial}w = f$  on  $X$ . Since  $\{s'_u \leq \rho \leq t'\}$  is a compact subset of  $\{\rho < t\}$ , this completes the proof.  $\square$

### 3. APPROXIMATION

Here we prove the following theorem:

**3.1. Theorem.** *Let  $\rho : X \rightarrow ]a, b[$  be a 1-corona such that  $H^1(X)$  is Hausdorff, and let  $a < t < b$ . Then each holomorphic function defined in a neighborhood of  $\{\rho \leq t\}$  can be approximated uniformly on  $\{\rho \leq t\}$  by holomorphic functions defined on  $X$ .*

We prove this by means of Grauert's bump method, similarly as in the case of strictly pseudoconvex manifolds (see, e.g., the proof of theorem 2.12.3 in [HL1]), although, in distinction to strictly pseudoconvex manifolds, here the  $\bar{\partial}$ -equation not always can be solved. This is possible, because for the bump method only the Hausdorffness (and not the vanishing) of all  $H^1(\{\rho < t\})$ ,  $a < t < b$ , is needed.

**3.2. Definition.** A collection

$$\left( \rho : X \rightarrow ]a, b[, \varphi : X \rightarrow ]a, b[, c, (U; z_1, \dots, z_n) \right)$$

will be called a **Grauert bump** if:

- (a)  $\rho : X \rightarrow ]a, b[$  and  $\varphi : X \rightarrow ]a, b[$  are strictly pseudoconvex coronas<sup>1</sup> such that  $\{\rho \leq c\} \subseteq \{\varphi \leq c\}$ ;
- (b)  $U$  is a relatively compact open subset of  $X$ , and  $z_1, \dots, z_n$  are holomorphic coordinates defined in a neighborhood of  $\bar{U}$  such that  $U$  is a ball with respect to these coordinates and  $\rho = \varphi$  in a neighborhood of  $X \setminus U$ .

**3.3. Lemma.** *Let*

$$\left( \rho : X \rightarrow ]a, b[, \varphi : X \rightarrow ]a, b[, c, (U; z_1, \dots, z_n) \right)$$

*be a Grauert bump. Then each holomorphic function defined in a neighborhood of  $\{\rho \leq c\}$  can be approximated uniformly on  $\{\rho \leq c\}$  by holomorphic functions defined in a neighborhood of  $\{\varphi \leq c\}$ .*

*Proof.* Let a holomorphic function  $f$  in a neighborhood of  $\{\rho \leq c\}$  be given. Choose  $\varepsilon > 0$  so small that

$$\left( \rho : X \rightarrow ]a, b[, \varphi : X \rightarrow ]a, b[, c + \varepsilon, (U; z_1, \dots, z_n) \right)$$

is still a Grauert bump and  $f$  is holomorphic in a neighborhood of  $\{\rho \leq c + \varepsilon\}$ . As  $\bar{U}$  is convex with respect to the coordinates  $z_1, \dots, z_n$  and the function  $\rho$  is plurisubharmonic in a neighborhood of  $\bar{U}$ , the set  $\bar{U} \cap \{\rho \leq c + \varepsilon\}$  is polynomially convex with respect to  $z_1, \dots, z_n$  (see, e.g., theorem 4.3.2 in [Ho] or theorem 2.7.1. in [HL1]). Therefore we can find a sequence of holomorphic functions  $u_\nu$  defined on a neighborhood of  $\bar{U}$ , which converges to  $f$  uniformly on  $\bar{U} \cap \{\rho \leq c + \varepsilon\}$ .

Since  $\rho = \varphi$  in a neighborhood of  $X \setminus U$ , we can find a relative compact open subset  $U'$  of  $U$  such that  $\rho = \varphi$  outside  $U'$ . Choose a  $\mathcal{C}^\infty$ -function  $\chi$  on  $X$  such that  $\chi \equiv 1$  in  $U'$  and  $\chi \equiv 0$  in a neighborhood of  $X \setminus U$ . Setting

$$v_\nu = \begin{cases} f + \chi(u_\nu - f) & \text{on } U \cap \{\rho \leq c + \varepsilon\}, \\ f & \text{on } (X \setminus U') \cap \{\varphi \leq c + \varepsilon\} = (X \setminus U') \cap \{\rho \leq c + \varepsilon\}, \\ u_\nu & \text{on } U', \end{cases}$$

we obtain a sequence of  $\mathcal{C}^\infty$ -functions  $v_\nu$  defined in a neighborhood of  $\{\varphi \leq c + \varepsilon\}$  such that

$$(3.1) \quad v_\nu \text{ converges to } f \text{ uniformly on } \{\rho \leq c + \varepsilon\},$$

$$(3.2) \quad \bar{\partial}v_\nu \text{ converges to zero in the Fréchet topology of } \mathcal{E}^{0,1}(\{\varphi < c + \varepsilon\}).$$

Choose  $t$  with  $a < t < c$  so close to  $a$  that  $\bar{U} \subseteq \{\varphi > t\} = \{\rho > t\}$ . Since, by theorem 2.5,  $\bar{\partial}\mathcal{E}^{0,0}(\{\varphi < c + \varepsilon\})$  is closed with respect to the Fréchet topology of

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<sup>1</sup>with the same  $X$  and  $]a, b[$  but different  $\rho$  and  $\varphi$

$\mathcal{E}^{0,1}(\{\varphi < c + \varepsilon\})$ , then, by the Banach open mapping theorem, it follows from (3.2) that there exists a sequence of functions  $w_\nu \in \mathcal{E}^{0,0}(\{\varphi < c + \varepsilon\})$  such that

$$(3.3) \quad \bar{\partial}w_\nu = \bar{\partial}v_\nu \text{ on } \{\varphi < c + \varepsilon\}$$

$$(3.4) \quad \text{and } w_\nu \text{ converges to zero uniformly on } \left\{t \leq \varphi \leq c + \frac{\varepsilon}{2}\right\}.$$

Then, in a neighborhood of  $\{\varphi \leq t\}$ ,

$$\bar{\partial}w_\nu = \bar{\partial}v_\nu = (u_\nu - f)\bar{\partial}\chi = 0.$$

Therefore, the functions  $w_\nu$  are holomorphic in a neighborhood of  $\{\varphi \leq t\}$ .

Since, for each sufficiently small  $\delta > 0$ ,  $\varphi : \{\varphi < t + \delta\} \rightarrow ]a, t + \delta[$  is a 1-corona, this implies that

$$(3.5) \quad \sup_{\varphi(\zeta) \leq t} |w_\nu(\zeta)| = \max_{\varphi(\zeta)=t} |w_\nu(\zeta)|.$$

Indeed, by the lemma of Morse (see, e.g., proposition 0.5 in Appendix B of [HL2]), we may assume that  $\varphi$  has only non-degenerate critical points. Now first let  $a < s' < s'' \leq t$  such that no critical point of  $\varphi$  lies on  $\{s' \leq \varphi \leq s''\}$ . Then, for  $s' \leq s \leq s''$ , locally, the surface  $\{\varphi = s\}$  is strictly convex with respect to appropriate holomorphic coordinates (as the boundary of  $\{\varphi < s\}$ ). Hence, for each point  $\xi \in \{\varphi < s\}$  sufficiently close to  $\{\varphi = s\}$ , there is a smooth complex curve  $L \subseteq X$  such that  $L \cap \{\varphi \leq s\}$  is compact and  $\xi \in L \cap \{\varphi < s\}$ , which yields

$$w_\nu(\xi) \leq \max_{\zeta \in L \cap \{\varphi=s\}} |w_\nu(\zeta)| \leq \max_{\varphi(\zeta)=s} |w_\nu(\zeta)|.$$

Therefore: whenever  $a < s' < s'' \leq t$  such that  $\varphi$  has no critical points on  $\{s' \leq \varphi \leq s''\}$ , then

$$\max_{s' \leq \varphi(\zeta) \leq s''} |w_\nu(\zeta)| = \max_{\varphi(\zeta)=s''} |w_\nu(\zeta)|.$$

As the critical points of  $\varphi$  are isolated, now (3.5) now follows by continuity.

It remains to set

$$h_\nu = v_\nu - w_\nu.$$

By (3.3) these functions  $h_\nu$  are holomorphic on a neighborhood of  $\{\varphi \leq c\}$ , and by (3.4) and (3.1), they converge to  $f$  uniformly on  $\{\rho \leq c\}$ .  $\square$

Theorem 3.1 now is an immediate consequence of the preceding lemma and the following lemma:

**3.4. Lemma.** *Let  $\rho : X \rightarrow ]a, b[$  be a strictly pseudoconvex corona, and let  $a < t < b$ . Then there exists a sequence*

$$\left( \rho_j : X \rightarrow ]a, b[, \varphi_j : X \rightarrow ]a, b[, c_j, (U_j; z_1^{(j)}, \dots, z_n^{(j)}) \right), \quad j \in \mathbb{N},$$

*of Grauert bumps such that*

- $\rho_0 = \rho$  and  $c_0 = t$ ,
- $\rho_{j+1} = \varphi_j$  for all  $j \in \mathbb{N}$ ,
- $X = \bigcup_{j=0}^{\infty} \{a < \rho_j \leq c_j\}$ .

The latter lemma follows easily from the fact that a  $\mathcal{C}^2$ -small perturbation of a strictly plurisubharmonic function is again such a function. We omit the proof, which is the same as in the case of a strictly pseudoconvex manifold (see, e.g., lemma 2.12.4 in [HL1] or lemma 12.3 in [HL2] and the subsequent remark).

## 4. PROOF OF THEOREM 1.2

In this section  $\rho : X \rightarrow ]a, b[$  is a 1-corona such that  $H^1(X)$  is Hausdorff. As already observed in the introduction, we only have to prove that conditions (A) and (B) are satisfied. To prove this, let two different points  $\xi, \eta \in X$  and a holomorphic tangential vector  $\theta$  of  $X$  at  $\xi$  be given. It is sufficient to show that:

- (A') There exists a holomorphic function  $f$  on  $X$  such that  $f(\xi) \neq f(\eta)$ .
- (B') There exists a sequence of holomorphic functions  $g_\nu$  on  $X$  such that  $\partial g_\nu(\xi)$  converges to  $\theta$ .

We may assume that  $\rho(\xi) \geq \rho(\eta)$ . Choose a system  $z = (z_1, \dots, z_n)$  of holomorphic coordinates defined in a neighborhood  $U$  of  $\xi$  such that  $\eta \notin U$ ,  $z(\xi) = 0$  and  $z(U)$  contains the unit ball. Let

$$F := -2 \sum_{j=1}^n \frac{\partial \rho(\xi)}{\partial z_j} z_j - \sum_{j,k=1}^n \frac{\partial^2 \rho(\xi)}{\partial z_j \partial z_k} z_j z_k$$

be the Levi polynomial with respect to  $z_1, \dots, z_n$  of  $\rho$  at  $\xi$ . As well known (see, e.g., [HL1]), then we can find  $0 < \varepsilon < 1/2$  with

$$\operatorname{Re} F \geq \rho(\xi) - \rho + 2\varepsilon|z|^2 \quad \text{on } \{|z| \leq 2\varepsilon\}.$$

It follows that

$$\operatorname{Re} F \geq \varepsilon^3 \quad \text{on } \{\rho \leq \rho(\xi) + \varepsilon^3\} \cap \{\varepsilon \leq |z| \leq 2\varepsilon\}.$$

Set  $h = e^{-F}$  on  $U$ . Then

$$h(\xi) = 1 \text{ and}$$

$$|h| \leq e^{-\varepsilon^3} < 1 \text{ on } \{\rho \leq \rho(\xi) + \varepsilon^3\} \cap \{\varepsilon \leq |z| \leq 2\varepsilon\}.$$

Choose a  $\mathcal{C}^\infty$ -function  $\chi$  on  $X$  with  $\chi \equiv 1$  in a neighborhood of  $\{|z| \leq \varepsilon\}$  and  $\chi \equiv 0$  in a neighborhood of  $X \setminus \{|z| < 2\varepsilon\}$ . Setting

$$f'_\nu = \begin{cases} h^\nu \chi & \text{on } \{|z| < 2\varepsilon\}, \\ 0 & \text{on } X \setminus \{|z| < 2\varepsilon\}, \end{cases}$$

we obtain a sequence  $f'_\nu$  of  $\mathcal{C}^\infty$ -functions on  $X$  with

$$(4.1) \quad f'_\nu(\xi) = 1 \text{ for all } \nu,$$

$$(4.2) \quad f'_\nu(\eta) = 0 \text{ for all } \nu \text{ and}$$

$$(4.3) \quad \bar{\partial} f'_\nu = h^\nu \bar{\partial} \chi \text{ converges to zero uniformly on } \{\rho \leq \rho(\xi) + \varepsilon^3\}.$$

Moreover, let  $\theta_j$  be the coefficients with  $\theta = \sum_{j=1}^n \theta_j \frac{\partial}{\partial z_j} \Big|_\xi$ . Then, setting

$$g'_\nu = \begin{cases} \left( \sum_{j=1}^n \theta_j z_j \right) f'_\nu & \text{on } \{|z| < 2\varepsilon\}, \\ 0 & \text{on } X \setminus \{|z| < 2\varepsilon\}, \end{cases}$$

we obtain a sequence  $g'_\nu$  of  $\mathcal{C}^\infty$ -functions on  $X$  with

$$(4.4) \quad \partial g'_\nu(\xi) = f'_\nu(\xi) \theta + \left( \sum_{j=1}^n \theta_j z_j(\xi) \right) \partial f'_\nu(\xi) = \theta \text{ for all } \nu \text{ and}$$

$$(4.5) \quad \bar{\partial} g'_\nu = \left( \sum_{j=1}^n \theta_j z_j \right) \bar{\partial} f'_\nu \text{ converges to zero uniformly on } \{\rho \leq \rho(\xi) + \varepsilon^3\}.$$

Since, by theorem 2.5,  $\overline{\partial}\mathcal{E}^{0,0}(\{\rho < \rho(\xi) + \varepsilon^3\})$  is closed in the Fréchet space  $\mathcal{E}^{0,1}(\{\rho < \rho(\xi) + \varepsilon^3\})$ , now, by (4.3), (4.5) and the Banach open mapping theorem, we can find sequences  $f''_\nu, g''_\nu \in \mathcal{E}^{0,0}(\{\rho < \rho(\xi) + \varepsilon^3\})$  which converge to zero uniformly on  $\{\rho \leq \rho(\xi) + \varepsilon^3/2\}$  such that

$$\overline{\partial}f''_\nu = \overline{\partial}f'_\nu \quad \text{and} \quad \overline{\partial}g''_\nu = \overline{\partial}g'_\nu \quad \text{for all } \nu.$$

Setting  $f'''_\nu = f'_\nu - f''_\nu$  and  $g'''_\nu = g'_\nu - g''_\nu$  we get holomorphic functions on  $\{\rho < \rho(\xi) + \varepsilon^3\}$  such that, by (4.1), (4.2) and (4.4),

$$(4.6) \quad f'''_\nu(\xi) \text{ converges to } 1,$$

$$(4.7) \quad f'''_\nu(\eta) \text{ converges to } 0 \text{ and}$$

$$(4.8) \quad \partial g'''_\nu(\xi) \text{ converges to } \theta.$$

*Proof of (A'):* By (4.6) and (4.7) we can find  $\nu_0$  so large that  $f'''_{\nu_0}(\xi) > 3/4$  and  $f'''_{\nu_0}(\eta) < 1/4$ . By theorem 3.1, now we can find a holomorphic function  $f$  on  $X$  such that  $|f - f'''| < 1/4$  on  $\{\rho \leq \rho(\xi)\}$ . As  $\xi, \eta \in \{\rho \leq \rho(\xi)\}$ , then  $f(\xi) > 1/2 > f(\eta)$ .

*Proof of (B'):* Again by theorem 3.1, we can find a sequence of holomorphic functions  $g_\nu$  on  $X$  such that  $g_\nu - g'''_\nu$  converges to zero uniformly on  $\{\rho \leq \rho(\xi) + \varepsilon^3/2\}$ . Since  $\xi$  is an inner point of  $\{\rho \leq \rho(\xi) + \varepsilon^3/2\}$ , then also  $\partial g_\nu(\xi) - \partial g'''_\nu(\xi)$  converges to zero. By (4.8) this implies that  $\partial g_\nu(\xi)$  converges to  $\theta$ .

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